Technical Notes

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Propagation of Weak Discontinuity through the Unsteady Flow of Thermally Conducting Dissociating Gases

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I. Introduction

WEAK discontinuities result from the singularities of the initial or boundary conditions of the flow. The propagation of weak discontinuities through different media has been studied by Thomas,[1,2] Kaul,[3] Nariboli,[4] Nariboli and Secrest,[5] Upadhyay,[6] and many others under the assumption that the medium in front of the propagating surface is uniform and at rest. In an unsteady flow of a perfect gas, Elcrat[7] studied the nonuniform propagation of sonic discontinuities and integrated the growth equations by transforming them into an equation along the bicharacteristics in the characteristic manifold. In this Note, we have studied the propagation of weak discontinuity through the unsteady flow of thermally conducting dissociating gases and obtained the critical time when a weak discontinuity grows into a shock wave.

II. Basic Equations

The equations governing the flow of an inviscid nonconducting gas with finite thermal conductivity for a dissociating gas are[8,9]:

$$\frac{\partial \sigma}{\partial t}+u_i\sigma,_i+\sigma u_{i,i}=0 \tag{1}$$

$$\sigma\left(\frac{\partial u_i}{\partial t}\right)+\sigma u_j u_{i,j}+p,_i=0 \tag{2}$$

$$\sigma T\left(\frac{\partial S}{\partial t}+u_i S,_i\right)-KT,_{ii}=0 \tag{3}$$

with the thermodynamical quantities related by

$$p=R(1+q)\sigma T \tag{4}$$

$$T\mathrm{d}S=3\mathrm{d}T+T\log\frac{1-q}{\sigma q^2}\,\mathrm{d}q-\frac{R(1+q)T}{\sigma}\mathrm{d}\sigma \tag{5}$$

and reaction rate equation given by

$$\frac{\partial q}{\partial t}+u_i q,_i=W \tag{6}$$

where t, T, K, S, and q are time, temperature, thermal conductivity, and degree of dissociation, respectively; a comma (,) in all of the equations in this Note indicates partial differentiation with respect to the coordinates x_i. The quantity W is given by

$$W=\frac{K_r(1+q)\sigma^2}{m_a^2}\left\{\frac{q_e^2}{1-q_e^2}(1-q^2)-q^2\right\}$$

where K_r is the recombination rate, m_a the atom mass, and q_e the equilibrium degree of dissociation given by

$$q_e=\left\{\frac{pT_d}{2p_dT}\left(\frac{T_v}{T}\right)^{1/2}\frac{\exp(T_d/T)}{1-\exp(-z)}+1\right\}^{-1/2}$$

Here p_d is defined by $p_d=\sigma_d RT_d$, where σ_d and T_d are the characteristic density and temperature for dissociation, respectively. Suppose there exists a singular surface $\Sigma(t)$ moving with normal speed G into a medium and let n_i be a unit normal to $\Sigma(t)$, pointing into the medium ahead. The flow and field parameters are continuous, while their first- and higher order derivatives are discontinuous across $\Sigma(t)$. Then as in Thomas[1,2],

$$[p]=[\sigma]=[u_i]=[q]=0$$

$$[u_{i,j}]=\lambda n_i n_j,\left[\frac{\partial u_i}{\partial t}\right]=-Gn_i$$

$$[p,_i]=\mu n_i,\left[\frac{\partial p}{\partial t}\right]=-G\mu$$

$$[\sigma,_i]=\zeta n_i,\left[\frac{\partial \sigma}{\partial t}\right]-G\zeta \tag{7}$$

where λ, μ, and ζ are suitable functions defined on the surface and quantity enclosed in a square bracket denotes the difference of that quantity across $\Sigma(t)$.

III. Velocity of Propagation of the Sonic Wave

Using the first order compatibility conditions in Eqs. (1), (2), and (6), we get

$$U\zeta=\sigma\lambda \tag{8a}$$

$$\sigma U\lambda=\mu \tag{8b}$$

$$U[q,_i]n_i=0 \tag{8c}$$

where $U=G-n_i n_i$. From the law of conservation of energy of the fluid for which $K\neq 0$ across the surface $\Sigma(t)$, we have

$$[T,_i]n_i=0 \tag{9}$$

Differentiating Eq. (4) and using the compatibility equations (7), we get

$$\mu=C^2\left\{\zeta+\frac{\sigma}{1+q}[q,_i]n_i\right\}\text{where } C^2=R(1+q)T$$

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The value of μ when substituted into Eq. (8b) gives

$$\sigma U\lambda = C^2\left\{\zeta + \frac{\sigma}{1+q}[q,_i]n_i\right\} \tag{10}$$

Equation (8c) implies either that 1) $U=0$ and $[q,_i]=0$, that 2) $U\neq 0$ and $[q,_i]=0$, or that 3) $U=0$ and $[q,_i]\neq 0$.

Case 1

When $U=0$ and $[q,_i]=0$, Eq. (10) gives $\zeta=0$ on $\Sigma(t)$. Then, it follows from Eq. (8a) that $\lambda=0$ and hence from Eq. (8b) that $\mu=0$. Equations (7) then show that $\Sigma(t)$ is not a surface of weak discontinuity as postulated. Hence $\zeta\neq 0$, that is, case 1 is not possible.

Case 2

$U\neq 0$ and $[q,_i]=0$. In this case Eq. (10) reduces to $\sigma U\lambda = C^2\zeta$ which on using Eq. (8a), yields $U^2=C^2$ since $\zeta\neq 0$. Obviously, $G=u_n+C$ where $u_n=u_in_i$, which shows that the sonic surface moves with the velocity (u_n+C) in a thermally conducting dissociating gaseous medium.

Case 3

When $U=0$ and $[q,_i]\neq 0$, it is evident from Eqs. (8a) and (8b) that both λ and μ become zero. Also Eq. (10) reduces to $C^2\{\zeta+[\sigma/(1+q)][q,_i]n_i\}=0$. But since $C^2\neq 0$, $\zeta=-[\sigma/(1+q)][q,_i]n_i$, which shows that ζ increases (decreases) with the increasing (decreasing) degree of dissociation.

IV. Growth Equation and Its Solution

Differentiating Eqs. (1) and (2) with respect to x_k, multiplying by n_k, summing with respect to k, and following the notations and symbols used in Thomas,[2] we get

$$U\frac{\delta\zeta}{\delta t} - (U^2\bar{\zeta} - \sigma U\bar{\lambda}_i n_i) + 2U\zeta(u_{i,j}n_in_j)_2 + 2\lambda U\left[\left(\frac{\partial\sigma}{\partial n}\right)_2 - \sigma\Omega\right] - 2\lambda U\zeta + Ug^{\alpha\beta}u_i\zeta_\alpha x_{i,\beta} = 0 \tag{11}$$

$$\sigma\frac{\delta\lambda}{\delta t} + (\bar{\mu} - \sigma U\bar{\lambda}_i n_i) - U\lambda\left(\frac{\partial\sigma}{\partial n}\right)_2 + \left(\frac{\partial u_i}{\partial t} + u_ku_{i,k}\right)_2\zeta n_i \times U\zeta\left(\frac{\partial u_k}{\partial n}\right)_2 n_k + U\zeta(u_{i,k}n_in_k)_2 + \sigma g^{\alpha\beta}u_k\lambda,_\alpha x_{k,\beta} = 0 \tag{12}$$

From Eqs. (5), (7), and (9) we get

$$T\left(\left[\frac{\partial S}{\partial t}\right] + u_i[S,_i]\right) = \frac{C^2U}{\sigma}\zeta \tag{13}$$

Evaluating the quantities in Eq. (3) across $\Sigma(t)$ and using Eq. (13), we have $[T_{ii}]=(C^2U/K)\zeta$.

Differentiating Eq. (4) twice with respect to x_i, using the second-order compatibility conditions, and substituting the value of $\bar{\mu}$ thus obtained into Eq. (12), we get

$$\sigma\frac{\delta\lambda}{\delta t} + C^2\left\{\bar{\zeta} + \frac{\sigma R(1+q)U}{K}\zeta + \frac{2}{1+q}\left([q,_i](\sigma,_i)_2 + (q,_i)_2\zeta n_i - [q,_i]\zeta n_i + \tfrac{1}{2}\sigma[q,_{ii}]\right)\right\}$$
$$+2R(1+q)(T,_i)_2\zeta n_i + 2\sigma R[q,_i](T,_i)_2 - \sigma U\bar{\lambda}_i n_i - U\lambda\left(\frac{\partial\sigma}{\partial n}\right)_2 + \left(\frac{\partial u_i}{\partial t} + u_ku_{i,k}\right)_2\zeta n_i + U\zeta\left(\frac{\partial u_k}{\partial n}\right)_2 n_k$$
$$+ U\zeta(u_{i,k}n_in_k)_2 + \sigma g^{\alpha\beta}u_k\lambda,_\alpha x_{k,\beta} = 0 \tag{14}$$

It is difficult to obtain a solution to the above equation. However, when $U=C$ and $[q,_i]=0$ (i.e., for the case 2), we proceed to solve it. As a consequence of Eq. (11), Eq. (14) can be written as

$$C\left(\frac{\delta\zeta}{\delta t} + g^{\alpha\beta}u_i\zeta,_\alpha x_{i,\beta}\right) + \sigma\left(\frac{\delta\lambda}{\delta t} + g^{\alpha\beta}u_k\lambda,_\alpha x_{k,\beta}\right) + \frac{\sigma R(1+q)C^3}{K}\zeta + 2R(1+q)(T,_i)_2\zeta n_i$$
$$+ \frac{2C^2}{1+q}\left((q,_i)_2\zeta n_i + \sigma[q,_{ii}]\right) + C\lambda\left(\frac{\partial\sigma}{\partial n}\right)_2 + \left(\frac{\partial u_i}{\partial t} + u_ku_{i,k}\right)_2\zeta n_i + C\zeta\left(\frac{\partial u_k}{\partial n}\right)_2 n_k$$
$$+3C(u_{i,k}n_in_k)_2\zeta - 2\lambda C\sigma\Omega - 2C\lambda\zeta = 0 \tag{15}$$

The difficulty in the interpretation of Eq. (15) because of the terms involving surface derivatives is removed if it is transformed into a differential equation along bicharacteristic curves. Thus, as suggested by Elcrat[7] Eq. (15) is transformed into

$$\frac{d\lambda}{dt} - \lambda^2 + A\lambda + B = 0 \tag{16}$$

where

$$A = \tfrac{1}{2}\left\{\frac{\sigma R(1+q)C^2}{K} + \frac{2R}{C}(1+q)(T,_i)_2 n_i + \frac{2C}{1+q}(q,_i)_2 n_i + \frac{C}{\sigma}\left(\frac{\partial\sigma}{\partial n}\right)_2 + \frac{1}{C}\left(\frac{\partial u_i}{\partial t} + u_ku_{i,k}\right)_2 n_i + \left(\frac{\partial u_k}{\partial n}\right)_2 n_k + 3(u_{i,k}n_in_k)_2 - 2C\Omega\right\}$$

$$B = \frac{C^2}{2(1+q)}[q,_{ii}]$$

This is the basic differential equation for the growth and decay of weak discontinuities associated with the wave surface $\Sigma(t)$ which propagates in a thermally conducting dissociating gas. Integration of Eq. (16) under the initial condition $\lambda=\lambda_0$ at $t=0$ gives

$$t = \frac{1}{(A^2+4B)^{1/2}}\log\left\{\frac{(2\lambda-A)-(A^2+4B)^{1/2}}{(2\lambda-A)+(A^2+4B)^{1/2}} \times \frac{(2\lambda_0-A)+(A^2+4B)^{1/2}}{(2\lambda_0-A)+(A^2+4B)^{1/2}}\right\} \tag{17}$$

When $\lambda\to\infty$ the weak discontinuity grows into a shock in the critical time t_c given by

$$t_c = \frac{1}{(A^2+4B)^{1/2}}\log\frac{(2\lambda_0-A)+(A^2+4B)^{1/2}}{(2\lambda_0-A)-(A^2+4B)^{1/2}} \tag{18}$$

In a nondissociating gas and assuming the medium in front of the shock at rest, Eq. (18) reduces to

$$t_c = -A\log\left(1+\frac{A}{\lambda_0}\right)$$

which agrees with the expression obtained by Upadhyay.[6]

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Evidence of Imbedded Vortices in a Three-Dimensional Shear Flow

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Introduction

THIS Note reports on flow phenomena encountered in an investigation originally conceived as a contribution to the experimental three-dimensional data base for use in turbulence modeling. These results suggest that turbulence modeling in three-dimensional mean flows may be considerably more difficult (compared to the two-dimensional case) than originally hoped, due to the possible existence of both steady and unsteady vortices and their interaction. The experiment was conducted in the Langley 20 in. Mach 6 Blowdown Air Tunnel[1] on the three-dimensional wedge model shown sketched on Fig. 1 (model run at zero pitch and yaw). The tip region was designed with a nearly constant surface pressure level and a nearly continuous surface curvature (Fig. 2) to avoid the formation of large tip vortices associated with pressure and surface curvature discontinuities.[2,3] The original object of the study was to measure the streamwise development of the three-dimensional wake region downstream of the tip region, where both $\partial u/\partial y$ and $\partial u/\partial z$ are present. The data were meant to represent a simple three-dimensional quasiparallel shear flow perhaps suitable to help begin the process of turbulence modeling in three-dimensional free flows (note pre-experiment idealization of expected flowfield shown on Fig. 1).

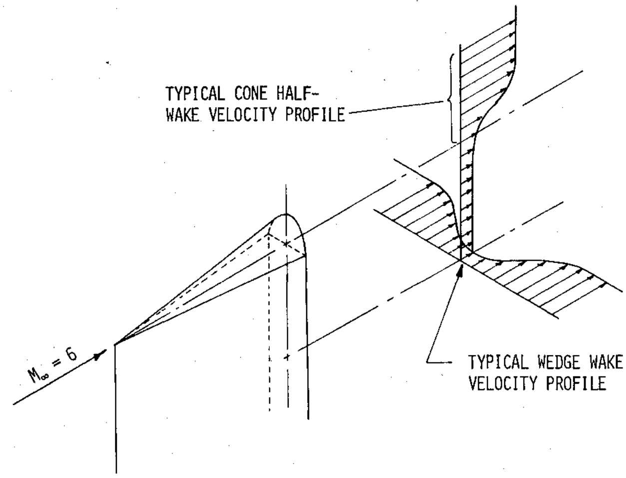

Fig. 1 Model and idealized velocity profiles in downstream wake.

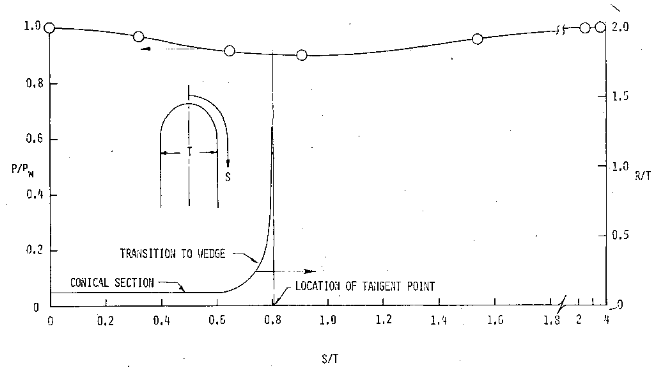

Fig. 2 Pressure distribution and radius of curvature around model near trailing edge.

Typical results of experimental pitot surveys are shown on Fig. 3. Instead of the rather simple flow sketched on Fig. 1, we obtained indication of a much more complex flow, with large deficits in longitudinal pitot pressure. Such deficits are usually associated with the core region of quasisteady longitudinal vortices. Vapor screen flow visualization results (Fig. 4) provide a further indication that the odd pitot distributions are indeed longitudinal vortices. These vortices 1) occur primarily in the tip region, becoming very weak as one approaches the two-dimensional zone away from the tip; and 2) evidently form in the vicinity of the wake neck. As an observation which may be of some technological importance, the overall wake thickness (and therefore turbulence entrainment rate) is increased by essentially 100%. (This suggests speculation that a fuel injector strut which is periodically necked down may provide somewhat faster mixing.) Previous work has already indicated that longitudinal vortices, once present, can considerably increase the mixing rate in free flows (e.g., "hypermixing" nozzles[4]). Trentacosta and Sforza[5] also found three-dimensional irregularities in the velocity profiles around the potential core of three-dimensional jets; however, their proposed explanation for the irregularities is the existence of a decaying ring vortex as opposing to the growing longitudinal vortices found in the present study.

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